CentraleSupélec

June 2019

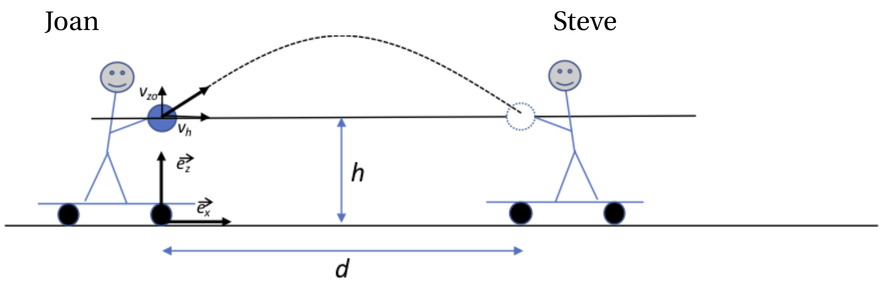
# Written exam in Mechanics

## 2 hours

This exam consists of three independent exercises. In addition the three parts of exercise 2 are independent. Each question carries at least one mark. Please make sure that you carefully write the steps of your solution method with the underlying assumptions, principles and laws you use at each step. For each question, make sure you specify your choice of system of coordinates (on a schematics of the system for example).

## Exercice 1. Joan et Steve play with a ball

Joan and Steve are standing on their skateboards which are directed in opposite directions. Both are at rest at the beginning of the experiment. The distance between them is $d$. Joan is holding a ball of mass $m$. Steve has a mass of M and Joan has a mass of $\mathrm{M} - m$. Hence Joan holding the ball is a system of mass M. The ball is subjected to a constant vertical gravity field $g$. The centres of mass of both players is assumed to stay at the same height $h$ during the experiment. It is assumed that the skateboards move along a single axis directed by unit vector $\overrightarrow{e_x}$. The ball only moves in the $(\overrightarrow{e_x}, \overrightarrow{e_z})$ plane. Skateboards are assumed to roll without sliding and without any energy dissipation. Air friction on the ball is neglected.




The ball leaves Joan's hands at height $h$, with horizontal velocity $v_h$ and vertical velocity $v_{zo}$ in the lab frame.

**Q. 1** *Give Joan's velocity after having thrown the ball.*

*Give the value of $v_{zo}$ the vertical component of the velocity of the ball so that the ball reaches Steve at height h ?*

**Q. 2** *Give the value of the sum of the kinetic and potential energy of the system made of the two players and the ball before and after Joan has thrown the ball (but before the ball reaches Steve). Comment on whether throwing the ball can be considered as an elastic collision.*

Steve catches the ball and it is assumed that he keeps the ball at height $h$ once received.

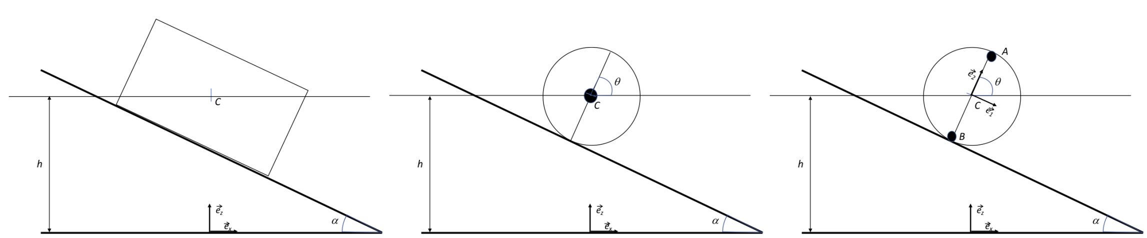
**Q. 3** *Give the value of Steve's velocity after having caught the ball.*
*Give the value of the sum of the kinetic and potential energy of the system made of the two players and the ball before and after Steve has caught the ball. Comment on whether catching the ball can be considered as an elastic collision.*

It is assumed now that Steve does not catch the ball. Moreover the ball is assumed to hit Steve at height $h$ and to bounce with a vertical component of the velocity $v_{zo}$ (the exact vertical component of the velocity applied by Joan when throwing the ball). The collision between the ball and Steve is assumed to be elastic.

**Q. 4** *Give the values of Steve's final velocity and of the final horizontal velocity of the ball.*

## Exercice 2. Downhill

Three different objects: a box and two different wheels, all having the same mass $m$ are positioned at rest at $t = 0$ on an angled slope with a tilt angle $\alpha$ and with their respective centres of mass at the same height $z = h$. Apart from vertical downwards gravity of magnitude $g$, these objects are only subjected to the reaction forces of the support. The friction coefficient $\mu$ may change values for the different objects (The magnitude of the friction-force $\vec{T}$ parallel to the slope and opposed to the motion is $\|\vec{T}\| \leq \mu \|\vec{N}\|$, with $\vec{N}$ the normal force applied by the slope on the object).



(a) (b) (c)

The box sliding without friction (Fig. 1a)

We assume that the rigid box slides without friction along the slope ($\mu = 0$).

**Q. 5** *Show that the normal reaction force* N *is constant and give its value as a function of* $\alpha$*,* $m$ *and* $g$*.*
*Show that the acceleration vector of the centre of mass is constant and give its direction and amplitude.*

**Q. 6** *Give the velocity vector of the center of mass* $\vec{v}_C$ *as a function of time* $t$.
*At what time* $t_b$ *does the centre of mass of the box reache* $z = 0$ *? Give* $v_{Cf}$*, the amplitude of the velocity vector at that time.*

The rolling wheel with concentrated mass at the centre (Fig. 1b)

We now consider a circular wheel of radius R rolling without sliding along the slope ($\mu > 0$). The total mass $m$ of the wheel is localised at its center. The rolling surface and any other part of the wheel are assumed to have no mass.

**Q. 7** *Using the conservation of energy, give* $v_C$ *the magnitude of the velocity vector of the centre of the mass as a function of its current height* $z$*,* $g$ *and* $h$.

**Q. 8** *At what time* $t_1$ *does the centre of the wheel reach* $z = 0$ *and what is the velocity* $v_{C1}$ *at that time ? Compare* $t_1$ *and* $t_b$ *and* $\vec{v}_{C1}$ *and* $\vec{v}_{Cf}$.

**Q. 9** *Give the values of the induced normal and tangential contact forces and make sure that the friction law is satisfied for* $\mu > 0$.

The rolling wheel with two opposite point-masses (Fig. 1c)

The second wheel has the same mass $m$ and same radius R as the previous one. However the mass is equally distributed at two points A and B located at the two ends of one diameter of the wheel ($m_A = m_B = m/2$). When the centre of the wheel is at $z = h$, B is assumed to be the contact point with the slope (see Fig. 1c). The unit vector $\vec{e_r}$ is defined as $\vec{e_r} = \overrightarrow{CA}/R$. $\theta(t)$ is the angle between $\vec{e_r}$ and the horizontal axis $\vec{e_x}$ with $\theta(0) = \theta_o = \pi/2 - \alpha$. Hence position vectors of A and B read:

$$\vec{r}_A = \vec{r}_C + R\vec{e_r} \qquad , \qquad \vec{r}_B = \vec{r}_C - R\vec{e_r}$$

**Q. 10** *Assuming rolling without sliding show that:*

$$\sin\alpha\, R\,(\theta(t) - \pi/2 + \alpha) = h - z(t)$$

*where* $z(t)$ *is the altitude of the centre of mass at time* $t$.
*Give the potential energy of the entire wheel including point-masses* A *and* B

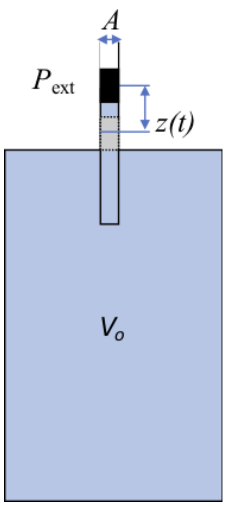
**Q. 11** *Give the velocity vector of points* A *and* B *as a function of* $\frac{dz}{dt}$, R, $\sin\alpha$ *and any convenient set of unit vectors amongst* $\{\vec{e_x}, \vec{e_z}, \vec{e_r}, \vec{e_\theta}\}$*. Give the kinetic energy of the entire wheel*

**Q. 12** *Give* $t_2$ *the time at which* C *the centre of the wheel reaches* $z = 0$*, and show that it is larger than* $t_1$ *and* $t_b$ *by a factor that does not depend on the radius of the wheel.*

**Q. 13** *Give the normal and tangential components of the reaction force as a fonction of $t$ and give a criteria on the friction coefficient $\mu$ to prevent sliding when rolling at all time $t$.*

*Compute the force applied by the remaining part of the wheel on the point-mass $A$.*

## Exercice 3. Ruchardt's experiment




Ruchardt's experiments aims at measuring the heat capacity ratio $\gamma$ of a given gas ($\gamma = \frac{C_p}{C_v}$). The experimental set-up consists of a gas container with a vertical outlet of circular cross-section with area $A = 1\ cm^2$. The tube is closed by a piston of mass $m = 10\ g$ having the same cross-section as the tube, with same area $A$. The piston is allowed to move freely along the tube with no friction. The gas is assumed to be an ideal gas with constant heat capacity ratio $\gamma$. It is assumed that $n$ the number of mol of gas in the container remains constant during the experiment - negligible leaks around the ball. The outside pressure $P_{ext}$ is assumed to be constant and equal to $10^5 Pa$.

It is assumed that the container and the ball do not allow any heat exchange with the outside. Gas transformations are assumed to be reversible. When the piston is at rest, the volume of gas is $V_o = 10l$, its temperature $T_o = 290K$ and its pressure is $P_o$. $z(t)$ denotes the vertical abscissa of the center of mass of the piston as e function of time. The inertial frame of reference is chosen such that at rest $z_{rest} = 0$. Changes in these state-variables are assumed to be small as compared to these reference values when the piston moves. The gravitational acceleration $g$ is assumed to be constant and equal to $10\ ms^{-2}$

**Q. 14** *The piston is assumed at rest at a given height without any applied action other than from gravity $g$, external pressure $P_{ext}$ and internal pressure $P_o$. Give $P_o$ as a function of $P_{ext}$, $m$, $g$, $A$ and $P_o$ (NB it is assumed that the mass density of the piston is much greater that the mass density of air). Conclude that, given the numerical values, $P_o \approx P_{ext}$.*

**Q. 15** *Assuming an adiabatic reversible transform, express $P$ as a function of $V$, $V_o$ and $P_o$ and find:*

$$\left.\frac{dP}{dV}\right|_{S,n}$$

*as a function of $\gamma$, $P$ and $V$.*

**Q. 16** *Assuming that $z$ is much smaller than $V_o/A$ show that the net force $\vec{F}$ applied on the piston reads:*

$$\vec{F} = -kz\vec{e_z} \qquad \textit{with} \qquad k = \gamma A^2 \frac{P}{V} \approx \gamma A^2 \frac{P_{ext}}{V_o}$$

**Q. 17** *Write the second Newton's Law of motion for the piston and deduce the ordinary differential equation satisfied by* $z(t)$.

**Q. 18** *At* $t = 0$*, the piston at* $z_{\text{est}}$ *is given an initial upward vertical velocity* $v_o = 0.2\ m/s$*. Give* $z$ *as a function of time* $t$*, initial velocity* $v_o$ *and a circular frequency* $\omega_o$ *to be given as a function of* $k$ *and* $m$.

**Q. 19** *Assuming that the piston oscillates with a period* $\mathrm{T}_o$*, give* $\gamma$ *as a function of* $\mathrm{T}_o$ *and the other parameters.*

*In the experiment we measure* $\mathrm{T}_o = 1.70 \pm 0.02\ s$*. What is the maximum amplitude reached by* $z$*,* $\Delta\mathrm{V}$ *and* $\Delta\mathrm{P}$*. Conclude on the obtained value for* $\gamma$ *and its accuracy.*

We want to confirm the obtained value using acoustic waves. It is recall that $c_o$ the speed of acoustic waves is given by:

$$c_o^{-2} = \left.\frac{d\rho}{d\mathrm{P}}\right|_{\mathrm{S},n}$$

**Q. 20** *For an ideal gas, show that:* $c_o$ *only depends on* $\gamma$*, the molar-mass* $m_n = 29\ \mathrm{g/mol}$*, the temperature* $\mathrm{T} = 290\mathrm{K}$ *and the ideal gas constant* $\mathrm{R} = 8.3\ \mathrm{J.mol^{-1}.K^{-1}}$*. What is the expected value of* $c_o$ *for the value of* $\gamma$ *measured in the Ruchardt's experiment ?*

*We use a tube of length* $\mathrm{L}$ *with two open ends, to confirm the value obtained for* $\gamma$*. What is the expected first frequency giving rise to harmonic standing pressure waves ?*